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# ON THE MOTION OF AN ARTIFICIAL EARTH SATELLITE

SEPTEMBER 1966

J. B. Frazer

Prepared for
DEPUTY FOR SURVEILLANCE AND CONTROL SYSTEMS
SPACE DEFENSE SYSTEM PROGRAM OFFICE (496L/474L)
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

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ABSTRACT

The periodic position and velocity perturbations of an artificial earth satellite are developed to the first order for all $J_n$, based on the theory by Brouwer as extended by Giacaglia. An explicit formulation is also provided for the subset $J_2$, $J_3$, $J_4$. The use of a position and velocity formulation circumvents the equatorial and circular orbit singularities found in conventional developments. The definition of the mean elements of the theory is modified to reduce the complexity of the position perturbations, as suggested by Merson's Theory, and the resulting changes to the secular terms are developed. In order to facilitate an empirical correction for drag, the observed mean motion is introduced as a mean element in place of the semi-major axis.

REVIEW AND APPROVAL

This Technical Report has been reviewed and is approved.

Thomas O. Wear

THOMAS O. WEAR, Colonel, USAF
Director, 496L/474L System Program Office
Deputy for Surveillance & Control Systems

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# SECTION I

## INTRODUCTION

The motion of a near-earth satellite is, in the first approximation, "Keplerian"; i.e., it conforms to certain empirical laws formulated in the seventeenth century by Kepler. In his Principia, Newton demonstrated that these laws described motion in an inverse-square force field. The force (or negative potential) function for such a field is of the form

$$U = \frac{\mu}{r} \tag{1}$$

where $r$ is the geocentric distance of the satellite and $\mu$ is the gravitational constant. The motion is conventionally described by six "orbital elements," $a$, $e$, $I$, $M$, $\omega$, $\Omega$. Kepler's second law states that the motion occurs along an ellipse with one focus at the primary. The inclination, $I$, and the argument of the ascending node, $\Omega$, serve to locate the plane containing this ellipse. The eccentricity, $e$, and the argument of perigee, $\omega$, define the shape of the ellipse and its orientation within the orbital plane. The semi-major axis, $a$, provides the scale of the ellipse as well as the orbital period; from Kepler's third law the period $P$ is given by

$$P = \frac{2\pi}{\mu^{\frac{1}{2}}} \; a^{3/2} \tag{2}$$

The location of the satellite within the ellipse is given by the mean anomaly, $M$, which measures the area swept out by the radius vector since perigee passage. In accordance with Kepler's first law the area swept out and hence the mean anomaly increases at a uniform rate; then

$$M(t) = M_o + n_o t \tag{3}$$

where $n_o$ is the mean motion, given by

$$n_o = \frac{2\pi}{P} = \mu^{\frac{1}{2}} a^{-3/2} \tag{4}$$

The mean anomaly must be converted to a geometric angle to be of use; the eccentric and true anomalies, E and v, are related to M by Kepler's equation

$$E - e \sin E = M \tag{5}$$

which must be solved by iteration, and by

$$\tan \frac{v}{2} = \left( \frac{1+e}{1-e} \right)^{\frac{1}{2}} \tan \frac{E}{2} \tag{6}$$

Following these computations, the geocentric position and velocity vectors $\underline{r}$ and $\underline{\dot{r}}$ are given by (See Figure 1),

$$\underline{r} = r \underline{U} \tag{7}$$

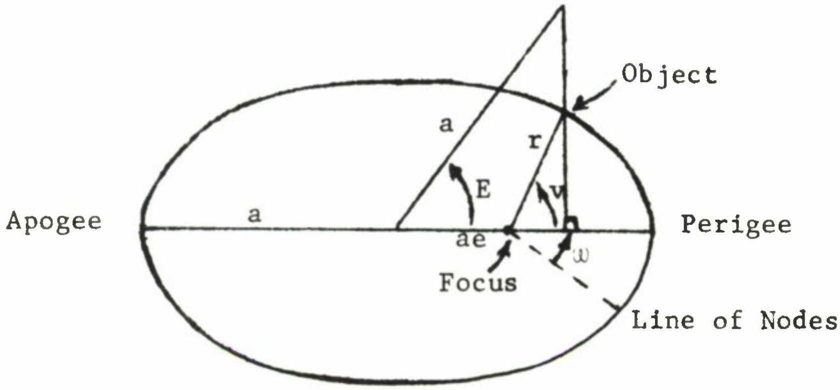
$$\underline{\dot{r}} = \dot{r} \underline{U} + r\dot{v}\underline{V} \tag{8}$$

where

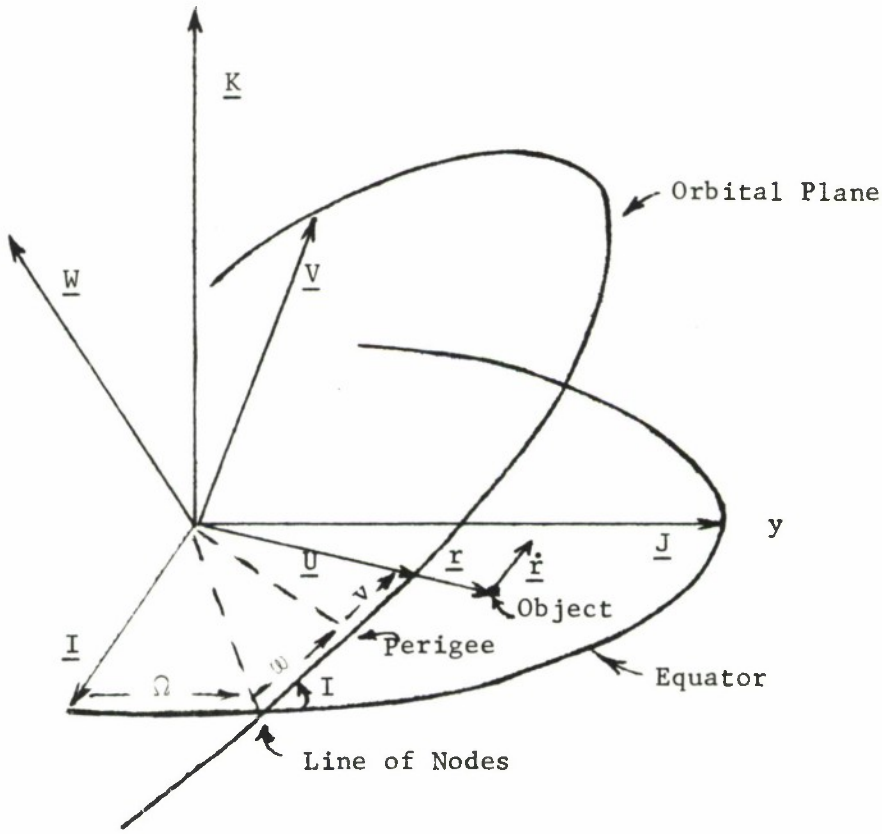
$$r = \frac{p}{1+e \cos v} \tag{9}$$

$$\dot{r} = \left(\frac{\mu}{p}\right)^{\frac{1}{2}} e \sin v \tag{10}$$

$$r\dot{v} = \left(\frac{\mu}{p}\right)^{\frac{1}{2}} (1+e \cos v) \tag{11}$$




POSITION IN THE ORBITAL PLANE




POSITION AND VELOCITY IN GEOCENTRIC INERTIAL COORDINATES

FIGURE 1

$$\underline{U} = \begin{bmatrix} \cos u \cos \Omega - \sin u \sin \Omega \cos I \\ \cos u \sin \Omega + \sin u \cos \Omega \cos I \\ \sin u \sin I \end{bmatrix} \tag{12}$$

$$\underline{V} = \begin{bmatrix} - \sin u \cos \Omega - \cos u \sin \Omega \cos I \\ - \sin u \sin \Omega + \cos u \cos \Omega \cos I \\ \cos u \sin I \end{bmatrix} \tag{13}$$

with

$$p = a\eta^2 \tag{14}$$

$$\eta = (1-e^2)^{\frac{1}{2}} \tag{15}$$

$$u = v+\omega \tag{16}$$

This simplified model cannot adequately represent the motion of a satellite; it must be augmented to include the effects of various "perturbing forces." These forces may be conservative, i.e., gravitational, or may affect the energy of the satellite's orbit, e.g., atmospheric drag and solar radiation pressure. More sophisticated models may be implemented via numeric integration or "special perturbations" techniques, which fall into three major categories:

(a) Integration in cartesian coordinates of accelerations resulting from all forces acting on satellite, to obtain position and velocity (Cowell's method).

(b) Integration in cartesian coordinates of accelerations resulting from perturbing forces only, to obtain deviations in position and velocity from a Keplerian orbit (Encke's method).

(c) Integration of the variations due to perturbing forces in the orbital elements of an "osculating" Keplerian orbit, i.e., the Keplerian orbit defined by the position and velocity of a satellite at each instant (Variation of Parameters).

These methods have a theoretical disadvantage, in that the accumulation of roundoff and truncation errors must eventually result in inadequate precision; it appears that in practice the length of arc is limited by the uncertainty in modeling non-conservative forces, which involve not only a complex and highly variable atmospheric structure, but also the configuration and orientation of the satellite. A more practical problem in some applications is that the integration must be carried from epoch to the most distant observation, regardless of whether useful data exists in the intervening period. In addition, the integrated orbit, whether in terms of coordinates or elements, provides little insight into the effects of the various forces operating on the satellite, so that it is difficult to identify and correct deficiencies in the model. Despite these disadvantages, special perturbations programs are widely employed for precision tracking where the frequency of data mitigates their relative inefficiency, or the cost is justified by the requirements for maximum precision. They are also extensively employed in feasibility studies and similar investigations, to avoid time consuming (and possibly impractical) analytic developments.

It is generally possible to obtain analytic expressions for the effects of the perturbing forces, to any desired precision and for any time span. Such "general perturbations" models are universally employed in routine cataloging systems, where a considerable number of satellites must be tracked with data that is sparsely distributed in time. In addition, general perturbations are usually employed in satellite geodesy. A considerable number of analytic theories have been developed for the conservative perturbing forces, i.e., the departure of the earth's gravity field from an inverse-square law and lunar and solar gravity.

For greater efficiency, semi-analytic theories are often employed for the luni-solar perturbations which are relatively small and of low frequency; the same approach is generally followed for the solar radiation pressure perturbations. Thus, for solar radiation pressure, the analytic development may be carried through a formal integration of the perturbations, but the results are left as a function of the limits of integration. These limits depend on the points at which the satellite enters and leaves the earth's shadow, which vary slowly with time. The evaluation of the perturbations proceeds n revolutions at a time, with the shadow limits reevaluated at each step.

Analytic models of the drag perturbation have been produced for simplified atmospheric models. The theory is complex, particularly when interactions with the earth's oblateness perturbations are considered.

As a result, empirical models are generally employed, with only the long term effects of drag considered. The results are generally satisfactory for high altitude objects, but there appears to be considerable merit in the development of a semi-analytic drag theory.

This paper deals only with perturbations due to the earth's gravity. In geodesy the gravity field is described in terms of a reference ellipsoid, a reasonably tractable figure which approximates the figure of a rotating fluid in equilibrium to about 1 part in $10^6$. The actual gravity at any point is shown in terms of a map of the elevation or depression of the "geoid" with respect to this ellipsoid; this "geoid" is an equipotential surface, i.e., a surface everywhere perpendicular to the local vertical. Before artificial satellites were launched the ellipsoid and "geoid" were determined from the reduction of direct gravity measurements and from astronomical determinations of the deviation of the local vertical from the local perpendicular to the ellipsoid. This "geoid" data is not employed in the theory of an artificial satellite, however. An analytic expansion for the potential is required; in spherical polar coordinates the generalized force (or negative potential) function is a series of Legendre polynomials and associated functions:

$$U = \frac{\mu}{r}\left\{1 - \sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n \left(J_n P_n (\sin\beta) + \sum_{m=1}^{n} J_{nm} P_{nm}(\sin\beta) \cos m (\lambda - \lambda_{nm})\right)\right\}$$

$$= \frac{\mu}{r}\left\{1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{r}\right)^n P_{nm}(\sin\beta)\,(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right\} \qquad (17)$$

where $a_e$ is the equatorial radius and $\beta$, $\lambda$ are the geocentric latitude and longitude of the satellite. The Legendre polynomials and associated functions are defined by

$$P_{nm}(x) = \frac{(1-x^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n = \frac{(1-x^2)^{m/2}}{2^n} \sum_{j=0}^{I(\frac{n-m}{2})} \frac{(-1)^j (2n-2j)!\, x^{n-m-2j}}{j!\,(n-j)!\,(n-m-2j)!} \tag{18}$$

where $I(\frac{n-m}{2})$ is the integer part of $(\frac{n-m}{2})$. The $P_n$ (or $P_{no}$) harmonics are "zonal," while the $P_{nm}$ harmonics are "tesseral." The largest coefficient is $J_2$; it is of order $10^{-3}$. The remaining coefficients do not exceed the second order, i.e., $10^{-6}$. In order to evaluate the effects of the harmonics, it is necessary to substitute orbital parameters for r, $\beta$, and $\lambda$. In general, a method of successive approximations must be employed, so that a series of perturbations of increasingly higher order arises, e.g.,

first order: $J_2$

second order: $J_2^2$, $J_n$, $J_{nm}$

third order: $J_2^3$, $J_2 J_n$, $J_2 J_{nm}$

Most general perturbation theories neglect periodic effects of the second order; the residual perturbations will then be on the order of 15 meters. However, under certain circumstances the perturbations due to higher order terms are amplified and must be included in a first order theory. If the potential function and its derivatives are expressed in terms of conventional orbital parameters, they will be found to have arguments of the form:

$$\left.\begin{matrix}\cos\\ \\ \sin\end{matrix}\right\} \left[(n-2p+q)M + (n-2p)\omega + m(\Omega-\lambda)\right]$$

where $n, m$ are the indices of the harmonic, $p$ ranges from 0 to $n$ (it is the parameter of a power series in sin I and cos I), and $q$ ranges from $-\infty$ to $\infty$ (it is the parameter of a power series in $e$, with the lowest power of $e$ being $e^{|q|}$). From a simple first order theory, it will be found that $M$, $\omega$, and $\Omega$ all increase linearly with time, so that when the perturbations are integrated divisors will arise of the form

$$(n-2p+q)n_o + (n-2p)\dot{\omega} + m(\dot{\Omega}-\dot{\lambda})$$

where $n_o$ is the perturbed mean motion. The perturbations are classified in terms of $n$, $m$, $p$, $q$ as follows:

(a) Secular terms $p = (n+q)/2$

$q = 0$

$m = 0$

These terms give rise to a linear increase in the elements $M$, $\omega$, $\Omega$, and are therefore computed to the second order in a first order theory (so that the theory is valid for about 10-20 days, after which the neglected third order terms exceed the second order). These terms only arise for even order zonal harmonics, i.e., $n = 2, 4, \ldots$ ; the values of the even zonal harmonics are generally based upon observed secular perturbations.

(b) Long period terms $p = (n+q)/2$

$$q \neq 0$$

$$m = 0$$

These terms have a divisor of the form $-q\dot{\omega}$ which is of order $10^{-3}$; second order forces therefore integrate into first order perturbations and must be included in a first order theory. There is no $J_2$ term of this form; if there were, a different type of solution would be required (there is a $J_2^2$ term of this form which reduces to $J_2$ on integration). There is a special case for the "critical inclination" $I \approx 63.4^o$, where $\dot{\omega}$ is of order $10^{-6}$, so that a "resonance" occurs. In this case, either a special solution is employed or the long period terms are not integrated, i.e., they are left in the form of secular rates. The long period perturbations for even zonal harmonics are factored by the eccentricity $e$ and can often be ignored; this is not the case for the odd zonals whose values are usually determined by analysis of observed long period variations in eccentricity and inclination.

(c) Short period terms $p \neq (n+q)/2$

These terms have a divisor containing $n_o$ so that the order of the perturbation remains unchanged upon integration. Therefore, only the $J_2$ terms need be included in a first order theory.

(d) Tesseral harmonic terms $m \neq 0$

There are two cases of interest here. For $p = (n+q)/2$ there are terms with frequencies near some multiple of the siderial rate, since to the zeroth order

$$(n-2p)\dot{\omega} + m(\dot{\Omega}-\dot{\lambda}) \approx -m\dot{\lambda}$$

The integration results in an increase on the order of $n_o/m\dot{\lambda}$ or about $16/m$ for near earth satellites. These terms contribute perturbations on the order of 100 meters, and decrease in importance as $n$ and $m$ increase.

For

$$(n-2p+q)n_o + (n-2p)\dot{\omega} + m(\dot{\Omega}-\dot{\lambda}) \approx 0$$

there is a resonance analagous to that holding near the critical inclination. The resonance will be in general larger for smaller values of $(n-2p)$ and $q$. The principle resonances thus arise for

$$m \approx n_o \qquad\qquad n \text{ odd}$$

where $n_o$ is expressed in revolutions/day. Obvious cases of potential near resonance are 24 and 12 hour satellites. High order resonances, e.g., $(n,m)$ of $(13,13)$, $(15,13)$, and $(15,14)$ have been reported for certain satellites with magnitudes on the order of 100-150 meters and periods of 2.5-5 days. Obviously, by going to a sufficiently high order harmonic, a resonance can be found for any near earth satellite. Fortunately, the net effect of these higher order terms is considered to approach the second order.

This paper does not deal with the tesseral harmonic perturbations. It is limited to some minor modifications of the secular terms developed by Brouwer(1) and extended by Giacaglia(2), and to a non-singular development of the long and short period perturbations due to the zonal harmonics. In the Brouwer and Giacaglia papers the perturbations of conventional elements are computed, which leads to singularities for low eccentricity or inclination. Lyddane(3) showed that the problem could be circumvented by either computing the perturbations to "non-singular" elements, e.g., $e \cos M$ and $e \sin M$, or by computing the perturbations in the position and velocity vectors. The former approach is employed in most general perturbations ephemeris generators(4), while the latter approach is employed in this paper. Although Garfinkel(5), Kozai(6), and Merson(7) have computed some of the position perturbations, velocity perturbations have generally been neglected.

The use of position and velocity perturbations has the advantage of revealing the "real" or observable effects of the perturbing forces; Merson(7), for example, has shown that some of the apparent perturbations of the orbital plane affect only the velocity vector and can be ignored in a tracking network based on positional data. In addition, a position and velocity theory appears to be somewhat more efficient than a "non-singular" elements theory, particularly when only positional data is used for element correction. The position and velocity theory has one disadvantage, in that the frequency of the long period terms becomes comparable to the short period terms, and they must be recomputed for each ephemeris point. (However, they are recomputed for each point in most theories, whether or not the computation is necessary.)

## SECTION II

## PERTURBATIONS IN POSITION AND VELOCITY

In developing the perturbations it is convenient to use the angular momentum unit vector $\underline{W}$, given by

$$\underline{W} = \underline{U} \times \underline{V} = \begin{bmatrix} \sin \Omega \sin I \\ - \cos \Omega \sin I \\ \cos I \end{bmatrix} \tag{19}$$

The perturbed position and velocity may be computed as

$$\underline{r} = (r + \delta r)\ (\underline{U} + \delta \underline{U}) \tag{20}$$

$$\dot{\underline{r}} = (\dot{r} + \delta \dot{r})\ (\underline{U} + \delta \underline{U}) + (r\dot{v} + \delta r\dot{v})\ (\underline{V} + \delta \underline{V}) \tag{21}$$

or the perturbations alone may be computed as

$$\delta \underline{r} = \delta r\ \underline{U} + r\ \delta \underline{U} \tag{22}$$

$$\delta \dot{\underline{r}} = \delta \dot{r}\ \underline{U} + \delta r\dot{v}\ \underline{V} + \dot{r}\ \delta \underline{U} + r\dot{v}\ \delta \underline{V} \tag{23}$$

ignoring second order terms.

The quantities $\delta \underline{U}$ and $\delta \underline{V}$ may be written as

$$\delta \underline{U} = \underline{V}\ (\delta u + \cos I\ \delta\Omega) + \underline{W}\ (\sin u\ \delta I - \cos u \sin I\ \delta\Omega) \tag{24}$$

$$\delta \underline{V} = -\underline{U}\ (\delta u + \cos I\ \delta\Omega) + \underline{W}\ (\cos u\ \delta I + \sin u \sin I\ \delta\Omega) \tag{25}$$

and hence we have

$$\begin{aligned} \delta \underline{r} = {} & \delta r\ \underline{U} + r\ (\delta u + \cos I\ \delta\Omega)\ \underline{V} \\ & + r\ (\sin u\ \delta I - \cos u \sin I\ \delta\Omega)\ \underline{W} \end{aligned} \tag{26}$$

$$\begin{aligned} \delta \dot{\underline{r}} = {} & \Big(\delta \dot{r} - r\dot{v}\ (\delta u + \cos I\ \delta\Omega)\Big)\ \underline{U} \\ & + \Big(\delta\ r\dot{v} + \dot{r}\ (\delta u + \cos I\ \delta\Omega)\Big)\ \underline{V} \\ & + \Big(\dot{r}\ (\sin u\ \delta I - \cos u \sin I\ \delta\Omega) \\ & + r\dot{v}\ (\cos u\ \delta I + \sin u \sin I\ \delta\Omega)\Big)\ \underline{W} \end{aligned} \tag{27}$$

For $\delta r$, $\delta \dot{r}$, and $\delta u$ we can either use Taylor series expansions in the conventional elements, or the ingenious equations of Izsak[(8)] with the Brouwer determining function $S$ written in Hill's canonical variables* $\{\dot{r}, G, H \mid r, u, \Omega\}$. The equations are

$$\delta r = r \frac{\delta a}{a} - a \cos v \, \delta e + \frac{a \sin v}{\eta} e \, \delta M \tag{28}$$

$$\delta \dot{r} = - \frac{\dot{r}}{2} \frac{\delta a}{a} + \left(\frac{\mu}{p}\right)^{1/2} \frac{(1+e \cos v)^2}{\eta^2} \sin v \, \delta e + \left(\frac{\mu}{p}\right)^{1/2} \frac{(1+e \cos v)^2}{\eta^3} \cos v \, e \, \delta M \tag{29}$$

$$\delta u = \frac{(2+e \cos v)}{\eta^2} \sin v \, \delta e + \frac{(1+e \cos v)^2}{\eta^3} \delta M + \delta \omega \tag{30}$$

or

$$\delta r = - \frac{\partial S}{\partial \dot{r}} \tag{31}$$

$$\delta \dot{r} = \frac{\partial S}{\partial r} \tag{32}$$

$$\delta u = - \frac{\partial S}{\partial G} \tag{33}$$

The second set of equations appears much simpler, and has been solved for the short period terms by Izsak. They are not so simple, however, when dealing with the long-period terms containing trigonometric functions of $\omega$.

The perturbation in $r\dot{v}$ may be computed from

$$\delta r\dot{v} = r\dot{v}\left( - \frac{1}{2} \frac{\delta a}{a} + \frac{\cos v \, (1+e \cos v) - e}{\eta^2} \delta e - \frac{\sin v \, (1+e \cos v)}{\eta^3} e \, \delta M \right) \tag{34}$$

---

* Brouwer and Giacaglia employ the Delauney canonical variables $\{L, G, H \mid l, g, h\}$, where

$$L = (\mu a)^{\frac{1}{2}} \qquad l = M$$
$$G = L\eta \qquad g = \omega$$
$$H = G \cos I \qquad h = \Omega$$

or

$$\delta r\dot{v} = \delta\left(\frac{G}{r}\right) = r\dot{v}\left(\frac{\delta G}{G} - \frac{\delta r}{r}\right) \tag{35}$$

using intermediate results for $\delta G$.

If the Taylor series expansions are employed, it is possible to rewrite Equation (27) using Equations (29), (30), and (34) as

$$\delta \underline{\dot{r}} = \left(\frac{\mu}{p}\right)^{1/2} \times$$

$$\Bigg\{ \Bigg( - \frac{e \sin v}{2} \frac{\delta a}{a} - \frac{(1+e \cos v) \sin v}{\eta^2} \delta e$$

$$- \frac{(1+e \cos v)^2}{\eta^3} \delta M - (1+e \cos v)(\delta\omega + \cos I \, \delta\Omega) \Bigg) \underline{U}$$

$$+ \Bigg( - \frac{1+e \cos v}{2} \frac{\delta a}{a} + \frac{\cos v + e}{\eta^2} \delta e + e \sin v (\delta\omega + \cos I \, \delta\Omega) \Bigg) \underline{V}$$

$$+ \Bigg( (\cos u + e \cos \omega) \delta I + (\sin u + e \sin \omega) \sin I \, \delta\Omega \Bigg) \underline{W} \Bigg\} \tag{36}$$

## SECTION III

## SHORT PERIOD PERTURBATIONS

Izsak[8] has already computed $\delta r$, $\delta \dot{r}$, and $\delta u$ as

$$\delta r = \frac{J_2 a_e^2}{4p} \left\{ \sin^2 I \cos 2u + (1-3\theta^2) \left( 1 + \frac{2\eta}{1+e\cos v} + \frac{e \cos v}{1+\eta} \right) \right\} \tag{37}$$

$$\delta \dot{r} = - \mu^{\frac{1}{2}} \frac{J_2 a_e^2}{4\, p^{5/2}} \left\{ 2 \sin^2 I \, (1+e\cos v)^2 \sin 2u \right.$$

$$\left. + (1-3\theta^2)\, e \sin v \left( \eta + \frac{(1+e\cos v)^2}{1+\eta} \right) \right\} \tag{38}$$

$$\delta u = - \frac{J_2 a_e^2}{8\, p^2} \left\{ 6(1-5\theta^2)\,(v-M) + 4(1-6\theta^2 + \frac{1-3\theta^2}{1+\eta})\, e \sin v \right.$$

$$+ (1-3\theta^2)\,(1-\eta) \sin 2v + 2\,(5\theta^2 - 2)\, e \sin (2u-v)$$

$$\left. + (7\theta^2 - 1) \sin 2u + 2\theta^2 e \sin (2u+v) \right\} \tag{39}$$

where

$J_2$ = coefficient of the second zonal harmonic

$a_e$ = earth's equatorial radius

$\theta$ = $\cos I$

From Brouwer's theory, with

$$\gamma_2 = \frac{J_2 a_e^2}{2\, a^2}$$

we have

$$\frac{\delta G}{G} = 3\,\frac{J_2 a_e^2}{4\,p^2}\,\sin^2 I\left\{\cos 2u + e\cos(2u-v) + \frac{e}{3}\cos(2u+v)\right\} \tag{40}$$

$$\cos I\,\delta\Omega = -\frac{J_2 a_e^2}{4\,p^2}\,\theta^2\left\{6\,(v-M+e\sin v) - 3\sin 2u - 3\,e\sin(2u-v) - e\sin(2u+v)\right\} \tag{41}$$

$$\delta I = \frac{J_2 a_e^2}{4\,p^2}\,\sin I\,\theta\left\{3\cos 2u + 3\,e\cos(2u-v) + e\cos(2u+v)\right\} \tag{42}$$

Hence

$$\delta u + \cos I\,\delta\Omega = \frac{J_2 a_e^2}{4\,p^2}\left\{(\sin^2 I)\left|\frac{\sin 2u}{2} + 2\,e\sin(2u-v)\right| - (1-3\theta^2)\left|3\,(v-M) + 2\,e\sin v\left(\frac{2+\eta}{1+\eta}\right) + \frac{1-\eta}{2}\sin 2v\right|\right\} \tag{43}$$

$$\sin u\,\delta I - \cos u\sin I\,\delta\Omega = \frac{J_2 a_e^2}{4\,p^2}\,\sin I\,\theta \times \left\{-3\sin u - 4\,e\sin\omega + 4\,e\cos u\sin v + 6\cos u\,(v-M)\right\} \tag{44}$$

$$\cos u\,\delta I + \sin u\sin I\,\delta\Omega = \frac{J_2 a_e^2}{4\,p^2}\,\sin I\,\theta\left\{3\cos u + 4e\cos(u+v) - 6\sin u\,(v-M)\right\} \tag{45}$$

$$\delta\, r\dot{v} = \mu^{\frac{1}{2}} \frac{J_2 a_e^2}{4\, p^{5/2}} (1+e \cos v) \times$$

$$\left\{ (\sin^2 I) \left( 2 \cos 2u + 2e \cos (2u-v) + e \cos 2u \cos v \right) - (1-3\theta^2) \left( \frac{3}{2} (1+\eta) + e \cos v \left(\frac{2+\eta}{1+\eta}\right) + \frac{1-\eta}{2} \cos 2v \right) \right\} \quad (46)$$

Substituting these terms in Equations (26) and (27) gives, after some simplification, the short period terms in Table I.

In some tracking programs the relationship of the mean semi-major axis and the secularly perturbed mean motion is taken from Kozai's equation 14[(6)]:

$$\bar{n}^2 \bar{a}^3 = \mu \left( 1 + 3 \frac{J_2 a_e^2}{4\, p^2} (1-3\theta^2)\, \eta \right) \quad (47)$$

Since

$$\bar{n} = n_o \left( 1 - 3 \frac{J_2 a_e^2}{2\, p^2} (1-3\theta^2)\, \eta + \ldots \right) \quad (48)$$

this implies

$$\bar{a} = a \left( 1 + 3 \frac{J_2 a_e^2}{4\, p^2} (1-3\theta^2)\, \eta + \ldots \right) \quad (49)$$

where $a$ is the mean semi-major axis of the Brouwer theory, defined by

$$n_o^2\, a^3 = \mu \quad (50)$$

TABLE I

SHORT PERIOD PERTURBATIONS

$$\delta \underline{r} = r\,\alpha_1 \Bigg\{ \Big[ \sin^2 I \,(1+e\cos v)\cos 2u + (1-3\theta^2)\Big(2\eta + (1+e\cos v)\Big(1+\frac{e\cos v}{1+\eta}\Big)\Big)\Big]\,\underline{U}$$

$$+ \Big[\sin^2 I \Big(\frac{\sin 2u}{2} + 2e\sin(2u-v)\Big) - (1-3\theta^2)\Big(3(v-M) + e\sin v\Big(\frac{4+2\eta+e\cos v}{1+\eta}\Big)\Big)\Big]\,\underline{V}$$

$$+ \sin I\theta \Big[-3\sin u - 4e\sin\omega + \cos u\,\big(4e\sin v + 6\,(v-M)\big)\Big]\,\underline{W}\Bigg\}$$

$$\delta \underline{\dot r} = -\left(\frac{\mu}{p}\right)^{\frac{1}{2}} \alpha_1 \Bigg\{ \Big[ \sin^2 I\,(1+e\cos v)\Big(\sin 2u\,\big(\tfrac{5}{2} + 2e\cos v\big) + 2e\sin(2u-v)\Big)$$

$$- (1-3\theta^2)\Big(e\sin v\Big(-\eta + \Big(\frac{3+2\eta}{1+\eta}\Big)(1+e\cos v)\Big) + 3\,(1+e\cos v)\,(v-M)\Big)\Big]\,\underline{U}$$

$$- \Big[\sin^2 I\Big(2\cos 2u + \frac{5}{2}\,e\,\big(\cos(2u-v) + \cos 2u\cos v\big) + e^2\,(2\cos 2\omega + \cos 2u\cos^2 v)\Big)$$

$$- (1-3\theta^2)\Big(3+e^2 + e\cos v\,\Big(3+\eta+\frac{1}{1+\eta}\Big) + e^2\sin^2 v + 3e\sin v\,(v-M)\Big)\Big]\,\underline{V}$$

$$- \sin I\theta\Big[(3+4e^2\cos^2 v)\cos u + 7e\cos(u+v) - 8e^2\sin v\sin\omega - 6\,(\sin u + e\sin\omega)\,(v-M)\Big]\,\underline{W}\Bigg\}$$

Where

$$\alpha_1 = \frac{J_2 a_e^2}{4\,p^2}$$

The use of $\bar{a}$ in place of $a$ in the computation of unperturbed $\underline{r}$ and $\dot{\underline{r}}$ requires that a compensating perturbation be applied to $\delta\underline{r}$ and $\delta\dot{\underline{r}}$:

$$\frac{\delta a}{a} = \frac{a-\bar{a}}{a} = -\,3\,\frac{J_2 a_e^2}{4\,p^2}\,(1-3\theta^2)\quad\eta \tag{51}$$

This results in the following changes in Table I:

(a) The term $2\eta$ in the $\underline{U}$ component of $\delta\underline{r}$ becomes $-\eta$

(b) The term $-\eta\, e \sin v$ in the $\underline{U}$ component of $\delta\dot{\underline{r}}$ becomes $+\frac{\eta}{2}\, e \sin v$

Merson has developed a theory(7) in which the short period position perturbations are minimized. His formulae (153-157) relate the osculating elements to conditions at the ascending node, to the second order in $J_2$. By eliminating all first order terms whose argument is a multiple of $\omega$, and terms factored by $u$, a set of first order pseudo short period terms is obtained. These result in the position perturbations given in Table II. (The first order terms factored by $u$ in Merson's theory are actually the sum of secular terms factored by $M$ and short period terms factored by $(u-M)$. They can therefore be included in the secular term computation, and we have taken this approach.)

Now, if we define $\epsilon_i$ to be Brouwer's mean elements updated for secular perturbations, and $\epsilon_i'$ to be "smoothed" mean elements updated

TABLE II

SHORT PERIOD PERTURBATIONS FOR "SMOOTHED" ELEMENTS

$$\delta \underline{r} = r\,\alpha_1 \Bigg\{ (1+e\cos v)\left[\sin^2 I \cos 2u + (1-3\theta^2)(1-e\cos v)\right]\underline{U}$$

$$+\left[\sin^2 I\left(\frac{\sin 2u}{2} + 2e\sin(2u-v)\right) + (1-3\theta^2)\,e^2 \sin v\cos v\right]\underline{V}$$

$$+\left[\sin I\,\theta\,(-4e\sin\omega + 4e\cos u\sin v)\right]\underline{W}\Bigg\}$$

$$\delta \underline{\dot{r}} = -\left(\frac{\mu}{p}\right)^{1/2}\alpha_1 \times$$

$$\Bigg\{ (1+e\cos v)\left[\sin^2 I\left(\sin 2u\left(\frac{5}{2} + 2e\cos v\right) + 2e\sin(2u-v)\right) - (1-3\theta^2)\,e\sin v\right]\underline{U}$$

$$-\left[\sin^2 I\left(2\cos 2u + \frac{5}{2}e\left(\cos(2u-v) + \cos 2u\cos v\right) + e^2\left(2\cos 2\omega + \cos 2u\cos^2 v\right)\right)\right.$$

$$\left. - (1-3\theta^2)\left(2(1+e\cos v) + e^2\sin^2 v\right)\right]\underline{V}$$

$$-\sin I\,\theta\left[6\cos u + 7e\cos(u+v) + 3e\cos\omega + 4e^2\left(\cos u\cos^2 v - 2\sin v\sin\omega\right)\right]\underline{W}\Bigg\}$$

Where

$$\alpha_1 = \frac{J_2 a_e^2}{4\,p^2}$$

for secular terms, then:

$$\underline{r}(\epsilon_i) + \delta \underline{r}_{Brouwer} = \underline{r}(\epsilon_i) + \sum_{i=1}^{6} \frac{\partial \underline{r}}{\partial \epsilon_i} (\epsilon_i' - \epsilon_i) + \delta \underline{r}_{Smoothed} \tag{52}$$

so that we have three simultaneous equations:

$$\sum_{i=1}^{6} \frac{\partial \underline{r}}{\partial \epsilon_i} \Delta\epsilon_i = \delta \underline{r}_{Brouwer} - \delta \underline{r}_{Smoothed} \tag{53}$$

where

$$\Delta\epsilon_i = \epsilon_i' - \epsilon_i$$

Using Equations (26), (28), (30) and Tables I and II,

$$\left( r \frac{\Delta a}{a} - a \cos v \, \Delta e + \frac{a \sin v}{\eta} e\Delta M \right) = r\, \alpha_1 \, (1-3\theta^2) \times \left[ \frac{1+3\eta+e^2}{2} + \left(\frac{2+\eta}{1+\eta}\right)\left(e \cos v + \frac{e^2}{2} \cos 2v\right) \right]$$

$$r\left( \left(2 \sin v + \frac{e}{2} \sin 2v\right) \frac{\Delta e}{\eta^2} + \frac{(1+e\cos v)^2}{\eta^3} \Delta M + \Delta\omega + \theta\Delta\Omega \right) = - r\, \alpha_1 (1-3\theta^2) \times \left[ 3\,(v-M) + \left(\frac{2+\eta}{1+\eta}\right)\left(2e \sin v + \frac{e^2}{2} \sin 2v\right) \right]$$

$$r\,(\sin u \, \Delta I - \cos u \sin I \Delta\Omega) = r\, \alpha_1 \sin I \theta \left[ 6 \cos u \, (v-M) - 3 \sin u \right] \tag{54}$$

If the $\Delta\epsilon$ are restricted to contain only constants or terms with $(v-M)$ as angular arguments these equations may be solved to yield:

$$\frac{\Delta a}{a} = 2\,\alpha_1\,(1-3\theta^2)\,\eta$$

$$\Delta e = -\,\alpha_1\,(1-3\theta^2)\left(\frac{2+\eta}{1+\eta}\right) e\eta^2$$

$$\Delta I = -\,3\,\alpha_1\,\sin I\theta$$

$$\Delta M = 0$$

$$\Delta\omega = -\,3\,\alpha_1\,(1-5\theta^2)\,(v-M)$$

$$\Delta\Omega = -\,6\,\alpha_1\,\theta\,(v-M) \tag{55}$$

The resulting changes in the velocity perturbations may be obtained from an equation similar to Equation (53); the signs of the $\Delta\epsilon_i$ should be changed to yield:

$$\delta\underline{\dot{r}}_{\text{Smoothed}} - \delta\underline{\dot{r}}_{\text{Brouwer}} = -\sum_{i=1}^{6} \frac{\partial\underline{\dot{r}}}{\partial\epsilon_i}\,\Delta\epsilon_i \tag{56}$$

With the aid of Equation (36)

$$\delta\underline{\dot{r}}_{\text{Smoothed}} - \delta\underline{\dot{r}}_{\text{Brouwer}} = -\left(\frac{\mu}{p}\right)^{\frac{1}{2}} \alpha_1\ \mathbf{X}$$

$$\Big\{(1-3\theta^2)\left[e\sin v\left(-\eta+\left(\frac{2+\eta}{1+\eta}\right)(1+e\cos v)\right)+3(1+e\cos v)(v-M)\right]\underline{U}$$

$$+\ (1-3\theta^2)\left[-1-e^2-e\cos v\left(\eta+\frac{2+\eta}{1+\eta}\right)-3e\sin v\,(v-M)\right]\underline{V}$$

$$-\ 3\sin I\theta\left[\cos u+e\cos\omega+2(\sin u+e\sin\omega)(v-M)\right]\underline{W}\Big\} \tag{57}$$

which, when added to $\delta\dot{\underline{r}}$ from Table I, yields the velocity perturbations in Table II.

These position and velocity perturbations represent a useful simplification of the equations in Table I; corresponding modifications to the $J_2$ and $J_2^2$ secular terms are derived in Section V.

## SECTION IV

## LONG PERIOD PERTURBATIONS

The first order long period perturbations for $J_n$, $n > 2$, have been developed by Giacaglia[2] and Garfinkel and McAllister[9]. Giacaglia's results have been employed since they are more readily truncated; the published paper unfortunately contains a number of errors:

(a) For the even harmonics, the upper index limit for $k$ in $\sum_{kij}$ is incorrect, since values of $i$ within the permitted range do not exist for $k=\frac{p}{2}$. Furthermore, for $k = 0$, $i$ can exceed $j$; in this case, $\delta_{p+1,\ 2j,\ 2i}$ vanishes.*

Therefore

$$\sum_{kij} = \sum_{k=0}^{\frac{p-2}{2}} \ \sum_{i=1}^{\min\left\{\begin{matrix} j \\ \frac{p-2k}{2}\end{matrix}\right\}} \ \sum_{j=1}^{\frac{p-2}{2}}$$

Similarly for the odd harmonics

$$\sum_{kij} = \sum_{k=0}^{\frac{p-1}{2}} \ \sum_{i=0}^{\min\left\{\begin{matrix} j \\ \frac{p-2k-1}{2}\end{matrix}\right\}} \ \sum_{j=0}^{\frac{p-3}{2}}$$

where $\frac{p-2j-1}{2}$ was incorrectly given as the upper index limit of $i$.

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* These practical problems reflect theoretical errors; i.e., $k = \frac{p}{2}$ terms are secular terms and also appear in the $\sum_{kj}$ portion of the disturbing function.

(b) In equation 21, the term

$$2\,\frac{1-e^2}{e^2}$$

should be

$$-\ 2j\,\frac{(1-e^2)}{e^2}$$

(c) Equation 22 must be multiplied by $H$.

(d) In equation 28, the factor

$$\frac{2j-1}{2i+1}$$

should be

$$\frac{2j+1}{2i+1}$$

(e) In equation 30 the term

$$+\ \frac{p-2k}{\sin^2 I}$$

should have a minus sign, and the factor

$$\frac{\cos\ (2i+1)g}{2i+1}$$

has been omitted.

Also, Giacaglia has the wrong signs for the third through fifth harmonic Brouwer coefficients ($k_n$, $\gamma_n$, and $A_{n.0}$) as functions of $J_n$. With these corrections, Giacaglia's results may be summarized as

$$\delta\varepsilon = S\ \Delta\varepsilon \tag{58}$$

where $S$ and $\Delta\varepsilon$ for even and odd $n$ are summarized in Table III.

TABLE III

GIACAGLIA'S LONG PERIOD PERTURBATIONS

$\delta\epsilon = S\,\Delta\epsilon$

Where, for Even $n$

$$S = -\frac{4}{3}(n-1)!\,\frac{J_n a_e^{n-2}}{J_2 p^{n-2}}\,(1-5\theta^2)^{-1}\sum_{k=0}^{\frac{n-2}{2}}\;\sum_{i=1}^{\min\left\{j,\,\frac{n-2k}{2}\right\}}\;\sum_{j=1}^{\frac{n-2}{2}}\frac{(-1)^{k+i}\,(2n-2k)!\,e^{2j}\sin^{n-2k} I}{2^{2(n-k+j)}\,k!\,(n-k)!\left(\frac{n-2k-2i}{2}\right)!\left(\frac{n-2k+2i}{2}\right)!\,(n-2j-1)!\,(j+i)!\,(j-i)!}$$

$\Delta a = 0$

$\Delta e = 2\,\frac{\eta^2}{e}\cos 2i\omega$

$\Delta I = -2\,\frac{\theta}{\sin I}\cos 2i\omega$

$\Delta M = 2j\,\frac{\eta^3}{e^2}\,\frac{\sin 2i\omega}{i}$

$\Delta\omega = \left[5-2n-10\theta^2(1-5\theta^2)^{-1} - 2j\,\frac{\eta^2}{e^2} + \frac{(n-2k)\theta^2}{\sin^2 I}\right]\frac{\sin 2i\omega}{i}$

$\Delta\Omega = \theta\left[10(1-5\theta^2)^{-1} - \frac{n-2k}{\sin^2 I}\right]\frac{\sin 2i\omega}{i}$

For Odd $n$

$$S = -\frac{4}{3}(n-1)!\,\frac{J_n a_e^{n-2}}{J_2 p^{n-2}}\,(1-5\theta^2)^{-1}\sum_{k=0}^{\frac{n-1}{2}}\;\sum_{i=0}^{\min\left\{j,\,\frac{n-2k-1}{2}\right\}}\;\sum_{j=0}^{\frac{n-3}{2}}\frac{(-1)^{k+i}\,(2n-2k)!\,e^{2j+1}\sin^{n-2k} I}{2^{2(n-k+j)}\,k!\,(n-k)!\left(\frac{n-2k-2i-1}{2}\right)!\left(\frac{n-2k+2i+1}{2}\right)!\,(n-2j-2)!\,(j+i+1)!\,(j-i)!}$$

$\Delta a = 0$

$\Delta e = \frac{\eta^2}{e}\sin(2i+1)\omega$

$\Delta I = -\frac{\theta}{\sin I}\sin(2i+1)\omega$

$\Delta M = -\frac{2j+1}{2i+1}\,\frac{\eta^3}{e^2}\cos(2i+1)\omega$

$\Delta\omega = -\left[5-2n-10\theta^2(1-5\theta^2)^{-1} - (2j+1)\,\frac{\eta^2}{e^2} + \frac{(n-2k)\theta^2}{\sin^2 I}\right]\frac{\cos(2i+1)\omega}{2i+1}$

$\Delta\Omega = -\theta\left[10(1-5\theta^2)^{-1} - \frac{n-2k}{\sin^2 I}\right]\frac{\cos(2i+1)\omega}{2i+1}$

It is convenient to reduce these equations to a common form. First, note that the summations can be written as

$$\sum_{i=1}^{\frac{n-2}{2}} \quad \sum_{j=i}^{\frac{n-2}{2}} \quad \sum_{k=0}^{\frac{n-2i}{2}} \qquad n \text{ even}$$

$$\sum_{i=0}^{\frac{n-3}{2}} \quad \sum_{j=i}^{\frac{n-3}{2}} \quad \sum_{k=0}^{\frac{n-2i-1}{2}} \qquad n \text{ odd}$$

Now introduce a variable $\lambda$

$$\begin{aligned} \lambda &= 2i && n \text{ even} \\ &= 2i+1 && n \text{ odd} \end{aligned} \tag{59}$$

so that

$$\begin{aligned} \lambda &= 2, 4, \ldots , (n-2) && n \text{ even} \\ &= 1, 3, \ldots , (n-2) && n \text{ odd} \end{aligned}$$

Similarly introduce

$$\begin{aligned} \mu &= 2j && n \text{ even} \\ &= 2j+1 && n \text{ odd} \end{aligned} \tag{60}$$

so that

$$\mu = \lambda, \lambda+2, \ldots , (n-2)$$

Also introduce

$$\nu = n-2k$$

with

$$\nu = \lambda, \lambda+2, \ldots , n \tag{61}$$

Note that $S$ is factored by $e \sin I$. Moving this factor from $S$ into the $\Delta\varepsilon$, and employing

$$\xi = \pi/2 - \omega \tag{62}$$

we obtain the results in Table IV.

It is now a simple matter to obtain

$$\delta r = -a\eta^2 \sin I \; S \left\{ \cos v \cos \lambda\xi + \frac{\mu}{\lambda} \sin v \sin \lambda\xi \right\} \tag{63}$$

$$\delta \dot{r} = \left(\frac{\mu}{p}\right)^{1/2} (1+e \cos v)^2 \sin I \; S \left\{ \sin v \cos \lambda\xi - \frac{\mu}{\lambda} \cos v \sin \lambda\xi \right\} \tag{64}$$

$$\delta r\dot{v} = \left(\frac{\mu}{p}\right)^{1/2} (1+e \cos v) \sin I \; S$$

$$\times \left\{ (1+e \cos v) \left( \cos v \cos \lambda\xi + \frac{\mu}{\lambda} \sin v \sin \lambda\xi \right) - e \cos \lambda\xi \right\} \tag{65}$$

$$\delta u + \cos I \; \delta\Omega = \sin I \; S \left\{ \left(2 \sin v + e \frac{\sin 2v}{2}\right) \cos \lambda\xi \right.$$

$$\left. - \left(2 \cos v + e \frac{\cos 2v}{2} + \frac{3e}{2}\right) \frac{\mu \sin \lambda\xi}{\lambda} + e\,(2n-5) \frac{\sin \lambda\xi}{\lambda} \right\} \tag{66}$$

$$\sin u \; \delta I - \cos u \sin I \; \delta\Omega = -e \, \theta \, S$$

$$\times \left\{ \sin u \cos \lambda\xi + \cos u \left( \nu - \frac{10 \sin^2 I}{1-5\theta^2} \right) \frac{\sin \lambda\xi}{\lambda} \right\} \tag{67}$$

$$\cos u \; \delta I + \sin u \sin I \; \delta\Omega = -e \, \theta \, S$$

$$\times \left\{ \cos u \cos \lambda\xi + \sin u \left( \frac{10 \sin^2 I}{1-5\theta^2} - \nu \right) \frac{\sin \lambda\xi}{\lambda} \right\} \tag{68}$$

from which, with Equations (26) and (27), the results in Table V are obtained.

## TABLE IV

## MODIFIED LONG PERIOD PERTURBATIONS

$$\delta\epsilon = S\,\Delta\epsilon$$

Where

$$S = -\frac{8\,(n-1)!}{3 \cdot 2^n}\;\frac{J_n\,a_e^{\,n-2}}{J_2\,p^{n-2}}\,(1-5\theta^2)^{-1}\sum_{\lambda,\mu,\nu}\frac{(-1)^{\frac{n-\nu}{2}}\,(n+\nu)!\,e^{\mu-1}\,\sin^{\nu-1} I}{2^{\mu+\nu}\left(\frac{n-\nu}{2}\right)!\left(\frac{n+\nu}{2}\right)!\left(\frac{\nu-\lambda}{2}\right)!\left(\frac{\nu+\lambda}{2}\right)!\,(n-\mu-1)!\left(\frac{\mu-\lambda}{2}\right)!\left(\frac{\mu+\lambda}{2}\right)!}$$

$$\Delta a = 0 \qquad\qquad \Delta M = -\frac{\eta^3}{e}\sin I\,\frac{\mu}{\lambda}\sin\lambda\xi$$

$$\Delta e = \eta^2 \sin I \cos\lambda\xi \qquad\qquad \Delta\omega = -e\sin I\left[5-2n-\mu\frac{\eta^2}{e^2}-10\theta^2\,(1-5\theta^2)^{-1}+\frac{\nu\theta^2}{\sin^2 I}\right]\frac{\sin\lambda\xi}{\lambda}$$

$$\Delta I = -e\theta\cos\lambda\xi \qquad\qquad \Delta\Omega = -e\sin I\theta\left[10\,(1-5\theta^2)^{-1}-\frac{\nu}{\sin^2 I}\right]\frac{\sin\lambda\xi}{\lambda}$$

With

$$\cos\lambda\xi = (-1)^{\lambda/2}\cos\lambda\omega,\ \lambda\ \text{even} \qquad\qquad \sin\lambda\xi = -(-1)^{\lambda/2}\sin\lambda\omega,\ \lambda\ \text{even}$$

$$= (-1)^{\left(\frac{\lambda-1}{2}\right)}\sin\lambda\omega,\ \lambda\ \text{odd} \qquad\qquad = (-1)^{\left(\frac{\lambda-1}{2}\right)}\cos\lambda\omega,\ \lambda\ \text{odd}$$

$$\lambda = 2,4,\cdots,(n-2),\quad n\ \text{even} \qquad\qquad \mu = \lambda,\ \lambda+2,\ \cdots,\ (n-2)$$

$$= 1,3,\cdots,(n-2),\quad n\ \text{odd} \qquad\qquad \nu = \lambda,\ \lambda+2,\ \cdots,\ n$$

## TABLE V

### LONG PERIOD PERTURBATIONS IN THE COORDINATES

$$\delta \underline{r} = r\,S \Bigg\{ - \sin I\,(1+e \cos v) \left[ \cos v \cos \lambda\xi + \sin v \frac{\mu \sin \lambda\xi}{\lambda} \right] \underline{U}$$

$$+ \sin I \left[ \left(2 \sin v + \frac{e \sin 2v}{2}\right) \cos \lambda\xi - \left(2 \cos v + \frac{e \cos 2v}{2} + \frac{3e}{2}\right) \frac{\mu \sin \lambda\xi}{\lambda} + e\,(2n-5) \frac{\sin \lambda\xi}{\lambda} \right] \underline{V}$$

$$- e\,\theta \left[ \sin u \cos \lambda\xi - \cos u \left( \frac{10 \sin^2 I}{1-5\theta^2} - \nu \right) \frac{\sin \lambda\xi}{\lambda} \right] \underline{W} \Bigg\}$$

$$\delta \underline{\dot{r}} = \left(\frac{\mu}{p}\right)^{\frac{1}{2}} S \Bigg\{ -\sin I\,(1+e \cos v) \left[ \sin v \cos \lambda\xi - (\cos v+e) \frac{\mu \sin \lambda\xi}{\lambda} + e\,(2n-5) \frac{\sin \lambda\xi}{\lambda} \right] \underline{U}$$

$$+ \sin I \left[ (\cos v+e) \cos \lambda\xi + \sin v\, \eta^2 \frac{\mu \sin \lambda\xi}{\lambda} + e^2 \sin v\,(2n-5) \frac{\sin \lambda\xi}{\lambda} \right] \underline{V}$$

$$-e\,\theta \left[ (\cos u+e \cos \omega) \cos \lambda\xi + (\sin u+e \sin \omega) \left( \frac{10 \sin^2 I}{1-5\theta^2} - \nu \right) \frac{\sin \lambda\xi}{\lambda} \right] \underline{W} \Bigg\}$$

It will be noted that $S$ vanishes for n=2. If this were not the case, it would be necessary to modify the basic unperturbed solution, since the resulting terms would have a $J_2/J_2$ coefficient and could not be treated as a perturbation. However, the magnitude of $J_2$ is such that the long period solution for $J_2$ must be carried to one higher approximation than for the other $J_n$, yielding perturbations with the coefficient $J_2^2/J_2$ or $J_2$. As given by Brouwer, these terms are:

$$\delta a = 0 \tag{69}$$

$$\delta e = S_2^2 \eta^2 \sin I \cos 2\omega \tag{70}$$

$$\delta I = -S_2^2 e \theta \cos 2\omega \tag{71}$$

$$\delta M = S_2^2 \frac{\eta^3}{e} \sin I \sin 2\omega \tag{72}$$

$$\delta\omega = - S_2^2 e \sin I \left[ \frac{1}{2} + \frac{1}{e^2} - \frac{\theta^2}{\sin^2 I} - \frac{10\theta^2}{(1-5\theta^2)(1-15\theta^2)} \right] \sin 2\omega \tag{73}$$

$$\delta\Omega = - S_2^2 e \sin I\theta \left( \frac{1}{\sin^2 I} + \frac{10}{(1-5\theta^2)(1-15\theta^2)} \right) \sin 2\omega \tag{74}$$

where

$$S_2^2 = \frac{1}{16} \frac{J_2 a_e^2}{p^2} \frac{e \sin I}{(1-5\theta^2)} (1-15\theta^2) \tag{75}$$

The additional perturbations are easily obtained:

$$\delta\underline{r}_2 = r\,\frac{1}{16}\,\frac{J_2\,a_e^2}{p^2}\,\frac{e\sin I}{(1-5\theta^2)}\,(1-15\theta^2)\left\{-\sin I\,(1+e\cos v)\cos(2u-v)\;\underline{U}\right.$$

$$+\sin I\left(2\sin(2u-v)+\frac{e\sin 2u}{2}\right)\;\underline{V}$$

$$\left.-\,e\theta\left(\sin(u-2\omega)-\frac{10\sin^2 I\cos u\sin 2\omega}{(1-5\theta^2)\,(1-15\theta^2)}\right)\;\underline{W}\right\} \tag{76}$$

$$\delta\dot{\underline{r}}_2 = \left(\frac{\mu}{p}\right)^{\frac{1}{2}}\,\frac{1}{16}\,\frac{J_2\,a_e^2}{p^2}\,\frac{e\sin I}{(1-5\theta^2)}\,(1-15\theta^2)$$

$$\times\left\{-\sin I\,(1+e\cos v)\left(\sin(2u-v)-\frac{e\sin 2\omega}{2}\right)\;\underline{U}\right.$$

$$+\sin I\left(\cos(2u-v)+e\cos 2\omega-\frac{e^2\sin v\sin 2\omega}{2}\right)\;\underline{V}$$

$$\left.-\,e\theta\left(\cos(u-2\omega)+e\cos\omega+\frac{10\sin^2 I\,(\sin u+e\sin\omega)\sin 2\omega}{(1-5\theta^2)\,(1-15\theta^2)}\right)\;\underline{W}\right\} \tag{77}$$

In most general perturbations formulations it is customary to employ only the $J_2^2$, $J_3$, and $J_4$ long period perturbations. For comparative purposes, these are given explicitly in Table VI. An evaluation yields

For $J_3$

$$\lambda = \mu = 1 \qquad \nu = 1,3$$

$$S_3 = +\frac{1}{2}\,\frac{J_3\,a_e}{J_2\,p}\,(1-5\theta^2)^{-1}\times\left\{\begin{matrix}4 & \nu = 1\\ -5\sin^2 I & \nu = 3\end{matrix}\right\} \tag{78}$$

$$\delta \underline{r}_3 = r \frac{J_3 a_e}{2J_2 p} \Big\{ \sin I \, (1+e \cos v) \sin u \; \underline{U}$$

$$+ \sin I \, (2+e \cos v) \cos u \; \underline{V}$$

$$+ e\theta \cos v \; \underline{W} \Big\} \qquad (79)$$

$$\delta \underline{\dot{r}}_3 = \left(\frac{\mu}{p}\right)^{\frac{1}{2}} \frac{J_3 a_e}{2J_2 p} \Big\{ - \sin I \, (1+e \cos v) \cos u \; \underline{U}$$

$$- \sin I \, (\sin u + e \sin \omega) \; \underline{V}$$

$$- e\theta \sin v \; \underline{W} \Big\} \qquad (80)$$

<u>For $J_4$</u>

$$\lambda = \mu = 2 \qquad \nu = 2,4$$

$$S_4 = + \frac{5}{16} \frac{J_4 a_e^2}{J_2 p^2} \frac{e \sin I}{(1-5\theta^2)} \times \begin{Bmatrix} 6 & \nu = 2 \\ -7 \sin^2 I & \nu = 4 \end{Bmatrix} \qquad (81)$$

$$\delta \underline{r}_4 = r \frac{5}{16} \frac{J_4 a_e^2}{J_2 p^2} \frac{e \sin I}{(1-5\theta^2)} (1-7\theta^2) \Big\{ - \sin I \, (1+e \cos v) \cos (2u-v) \; \underline{U}$$

$$+ \sin I \left| 2 \sin (2u-v) + \frac{e \sin 2u}{2} \right| \underline{V}$$

$$- e\theta \left| \sin (u-2\omega) - \frac{2 \sin^2 I \cos u \sin 2\omega}{(1-5\theta^2)(1-7\theta^2)} \right| \underline{W} \Big\}$$

(82)

$$\delta \dot{\underline{r}}_4 = \left(\frac{\mu}{p}\right)^{\frac{1}{2}} \frac{5}{16} \frac{J_4 a_e^2}{J_2 p^2} \frac{e \sin I}{(1-5\theta^2)} (1-7\theta^2)$$

$$\times \left\{ -\sin I (1+e \cos v) \left(\sin (2u-v) - \frac{e \sin 2\omega}{2}\right) \underline{U} \right.$$

$$+ \sin I \left(\cos (2u-v) + e \cos 2\omega - \frac{e^2 \sin v \sin 2\omega}{2}\right) \underline{V}$$

$$\left. - e\theta \left(\cos (u-2\omega) + e \cos \omega + \frac{2 \sin^2 I (\sin u + e \sin \omega) \sin 2\omega}{(1-5\theta^2)(1-7\theta^2)}\right) \underline{W} \right\} \qquad (83)$$

From which the equations in Table VI are easily obtained.

## TABLE VI

## $J_2$ - $J_4$ LONG PERIOD PERTURBATIONS

$$
\begin{aligned}
\delta \underline{r} = r \Bigg\{ & \sin I\,(1+e\cos v)\left(\alpha_2 \sin u - \alpha_3 \cos(2u-v)\right) && \underline{U} \\
& + \sin I \left(\alpha_2 (2+e\cos v)\cos u + \alpha_3 \left(2\sin(2u-v) + \frac{e \sin 2u}{2}\right)\right) && \underline{V} \\
& + e\theta \left(\alpha_2 \cos v - \alpha_3 \sin(u-2\omega) + \alpha_4 \cos u \sin 2\omega\right) && \underline{W} \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\delta \underline{\dot{r}} = \left(\frac{\mu}{p}\right)^{\frac{1}{2}} \Bigg\{ & - \sin I\,(1+e\cos v)\left(\alpha_2 \cos u + \alpha_3 \left(\sin(2u-v) - \frac{e \sin 2\omega}{2}\right)\right) && \underline{U} \\
& - \sin I \left(\alpha_2 (\sin u + e \sin \omega) - \alpha_3 \left(\cos(2u-v) + e\cos 2\omega - \frac{e^2 \sin v \sin 2\omega}{2}\right)\right) && \underline{V} \\
& - e\theta \left(\alpha_2 \sin v + \alpha_3 (\cos(u-2\omega) + e\cos\omega) + \alpha_4 (\sin u + e \sin\omega)\sin 2\omega\right) && \underline{W} \Bigg\}
\end{aligned}
$$

Where

$$
\alpha_2 = \frac{1}{2}\frac{J_3 a_e}{J_2 p} \qquad \alpha_3 = \frac{e \sin I}{(1-5\theta^2)}\left[\frac{1}{16}\frac{J_2 a_e^2}{p^2}(1-15\theta^2) + \frac{5}{16}\frac{J_4 a_e^2}{J_2 p^2}(1-7\theta^2)\right]
$$

$$
\alpha_4 = \frac{10}{16}\frac{e \sin^3 I}{(1-5\theta^2)^2}\left(\frac{J_2 a_e^2}{p^2} + \frac{J_4 a_e^2}{J_2 p^2}\right)
$$

## SECTION V

## SECULAR TERMS

The results of Brouwer[1] and Giacaglia[2] are given in Table VII, where

$$S = \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{(n-1)!}{2^{2n}} \frac{J_n a_e^n}{p^n} \sum_{k=0}^{n/2} \sum_{j=0}^{\frac{n-2}{2}} \frac{(-1)^k (2n-2k)!\, e^{2j} \sin^{n-2k} I}{2^{2(j-k)} k!(n-k)!\left(\frac{n-2k}{2}!\right)^2 (j!)^2 (n-2j-1)!} \tag{84}$$

In most practical applications it is the parameter $n$ rather than $a$ which is determined. From the relationship

$$\frac{1}{1+\alpha} = 1 - \alpha + \alpha^2 + \dots$$

where $(1+\alpha)$ represents the terms factored by $n_o$ in the equation for $\bar{n}$ we obtain to the second order:

$$\begin{aligned} n_o &= \bar{n}\,(1-\alpha+\alpha^2+\dots) \\ &= \bar{n}\left\{1 + \frac{3}{4}\frac{J_2 a_e^2}{p^2}\eta\,(1-3\theta^2) + \frac{45}{128}\frac{J_4 a_e^4}{p^4}\eta\, e^2\left[3-30\theta^2+35\theta^4\right]\right. \\ &\qquad \left. - \frac{3}{128}\frac{J_2^2 a_e^4}{p^4}\eta\left[10-8\eta-25e^2+(-60+48\eta+90e^2)\theta^2+(130-72\eta-25e^2)\theta^4\right]\right\} \\ &= \bar{n}\left\{1 - S\eta\left(2j\,\frac{1-e^2}{e^2} - 3\right) + \left[J_2^2 \text{ term above}\right]\right\} \end{aligned} \tag{85}$$

TABLE VII

SECULAR TERMS ACCORDING TO BROUWER AND GIACAGLIA

$$\dot{M} = \overline{n} = n_o \left\{ 1 - \frac{3}{4} \frac{J_2 a_e^2}{p^2} \eta (1-3\theta^2) - \frac{45}{128} \frac{J_4 a_e^4}{p^4} \eta e^2 \left[ 3 - 30\theta^2 + 35\theta^4 \right] \right.$$

$$\left. + \frac{3}{128} \frac{J_2^2 a_e^4}{p^4} \eta \left[ 10 + 16\eta - 25e^2 + (-60 - 96\eta + 90e^2) \theta^2 + (130+144\eta-25e^2)\theta^4 \right] \right\}$$

$$= n_o \left[ 1 + S\eta \left(2j \frac{1-e^2}{e^2} - 3\right) + \left( J_2^2 \text{ term above} \right) \right]$$

$$\dot{\omega} = n_o \left\{ - \frac{3}{4} \frac{J_2 a_e^2}{p^2} (1-5\theta^2) - \frac{15}{128} \frac{J_4 a_e^4}{p^4} \left[ 12+9e^2 - (144+126e^2) \theta^2 + (196+189e^2) \theta^4 \right] \right.$$

$$\left. + \frac{3}{128} \frac{J_2^2 a_e^4}{p^4} \left[ -10+24\eta-25e^2 + (-36-192\eta+126e^2) \theta^2 + (430+360\eta-45e^2) \theta^4 \right] \right\}$$

$$= n_o \left( S \left[ 1 - 2n - 2j \frac{1-e^2}{e^2} + \frac{(n-2k)\theta^2}{\sin^2 I} \right] + \left[ J_2^2 \text{ term above} \right] \right)$$

$$\dot{\Omega} = n_o \theta \left\{ - \frac{3}{2} \frac{J_2 a_e^2}{p^2} - \frac{15}{32} \frac{J_4 a_e^4}{p^4} (2+3e^2)(3-7\theta^2) + \frac{3}{32} \frac{J_2^2 a_e^4}{p^4} \left[ 4+12\eta-9e^2 + (-40-36\eta+5e^2) \theta^2 \right] \right\}$$

$$= - n_o \left[ \theta S \left( \frac{n-2k}{\sin^2 I} \right) + \left( J_2^2 \text{ term above} \right) \right]$$

If $\bar{n}$ is substituted for $n_o$ in the equations for $\dot{\omega}$ and $\dot{\Omega}$ in Table VII, the complexity of the $J_2^2$ terms is reduced; they become:

For $\dot{\omega}$

$$\bar{n}\ \frac{3}{128}\ \frac{J_2^2 a_e^4}{p^4}\left[-10-25e^2 + (-36+126e^2)\theta^2 + (430-45e^2)\theta^4\right] \tag{86}$$

For $\dot{\Omega}$

$$\bar{n}\theta\ \frac{3}{32}\ \frac{J_2^2 a_e^4}{p^4}\left[4-9e^2 + (-40+5e^2)\theta^2\right] \tag{87}$$

The semi-major axis may be computed from the relationship

$$a = \mu^{1/3}\ n_o^{-2/3} = \mu^{1/3}\ \bar{n}^{-2/3}\ (1+2/3\ \alpha\ -\ 1/9\ \alpha^2 \ldots)$$

from which we obtain

$$\begin{aligned} a = \mu^{1/3}\ \bar{n}^{-2/3} \Bigg\{ & 1 - 1/2\ \frac{J_2 a_e^2}{p^2}\ \eta\ (1-3\theta^2) \\ & + \frac{1}{64}\ \frac{J_2^2 a_e^4}{p^4}\ \eta\ \Big[10+12\eta-25e^2 + (-60-72\eta+90e^2)\theta^2 \\ & \qquad\qquad +(130+108\eta-25e^2)\theta^4\Big] \\ & - \frac{15}{64}\ \frac{J_4 a_e^4}{p^4}\ \eta\ e^2\ \left[3-30\theta^2 + 35\theta^4\right] \Bigg\} \end{aligned}$$

$$= \mu^{1/3}\ \bar{n}^{-2/3} \left\{ 1+s\eta \left(\frac{4j}{3}\ \frac{1-e^2}{e^2} - 2\right) + \left[J_2^2 \text{ term above}\right] \right\} \tag{88}$$

Note that p, which is a function of a, enters into these equations. If we employ

$$\tilde{p} = \mu^{1/3} \bar{n}^{-2/3} (1-e^2)$$

$$= p\left(1 + \frac{1}{2} \frac{J_2 a_e^2}{p^2} \eta (1-3\theta^2) + \ldots\right)$$

in the equations in Table VII five $J_2^2 \eta$ terms are added to Equations (86), (87), and (88). It is therefore more efficient to compute $\tilde{p}$ from $\bar{n}$ and $p^{-2}$ from

$$p^{-2} = \tilde{p}^{-2}\left(1 + \frac{J_2 a_e^2}{p^2} \eta (1-3\theta^2) + \ldots\right) \tag{89}$$

for use in the $J_2$ terms. Either $\tilde{p}$ or p may be used in the $J_2^2$ terms, since the error in using $\tilde{p}$ is of the third order. If, however, Equation (88) is only carried to the first order, it may be entered with $\tilde{p}$, and p computed from a, so that Equation (89) is not required.

If the Kozai $\bar{a}$ is to be employed as an element, then Equation (88) is replaced by

$$\bar{a} = a\left(1 + \frac{3}{4} \frac{J_2 a_e^2}{p^2} \eta (1-3\theta^2)\right)$$

$$= \mu^{1/3} \bar{n}^{-2/3}\left(1 + \frac{J_2 a_e^2}{4 p^2} \eta (1-3\theta^2)\right) \tag{88a}$$

to the first order. The use of $\bar{n}$ and

$$\bar{p} = \bar{a}\,(1-e^2)$$

in the equations in Table VII would again lead to additional terms in $\eta$ in Equations (86) and (87). The preferred approach is therefore to correct $\bar{p}$ using

$$p^{-2} = \bar{p}^{-2}\left(1 + \frac{3}{2}\,\frac{J_2 a_e^2}{p^2}\,\eta\,(1-3\theta^2)\right) \tag{89a}$$

A rather more complicated approach is employed in the GEPERS routine(4) developed for use in the SPACETRACK system. The parameters employed include $\bar{a}$ and $\bar{p}$, with a mean motion parameter $\tilde{n}$ defined by

$$\tilde{n}^2\,\bar{a}^3 = \mu\left(1 + \frac{3}{4}\,\frac{J_2 a_e^2}{\bar{p}^2}\,\eta\,(1-3\theta^2)\right) \tag{90}$$

which is interpreted to mean

$$\begin{aligned}
\tilde{n} &= n_o\left[1 - \frac{3}{4}\,\frac{J_2 a_e^2}{\bar{p}^2}\,\eta\,(1-3\theta^2)\right] \\
&= n_o\left[1 - \frac{3}{4}\,\frac{J_2 a_e^2}{p^2}\,\eta\,(1-3\theta^2)\right. \\
&\qquad \left. + \frac{3}{128}\,\frac{J_2^2 a_e^4}{p^4}\,\eta\,(48\eta - 288\eta\theta^2 + 432\eta\theta^4)\right]
\end{aligned}$$

(where either p or $\bar{p}$ may be used in terms of order $J_2^2$.) If this equation is subtracted from the equation for $\bar{n}$ in Table VII and $\tilde{n}$ is subtracted for $n_o$ in the second order terms we obtain:

$$\bar{n} = \tilde{n} \left\{ 1 + \frac{3}{128} \frac{J_2^2 a_e^4}{\bar{p}^4} \eta \left[ 10-32\eta-25e^2 + (-60+192\eta+90e^2)\theta^2 + (130-288\eta-25e^2)\theta^4 \right] - \frac{45}{128} \frac{J_4 a_e^4}{\bar{p}^4} \eta\, e^2 \left[ 3-30\theta^2 + 35\theta^4 \right] \right\} \tag{91}$$

The expressions for $\dot{\omega}$ and $\dot{\Omega}$ in GEPERS employ $\tilde{n}$ and $\bar{p}$ in place of $\bar{n}$ and p; $\tilde{n}$ agrees with $\bar{n}$ to the first order, but the use of $\tilde{p}$ makes it necessary to modify Equations (86) and (87). The results may be obtained from Equation (86a) and are:

For $\dot{\omega}$

$$\tilde{n} \; \frac{3}{128} \frac{J_2^2 a_e^4}{p^4} \left[ -10-48\eta-25e^2 + (-36+384\eta+126e^2)\theta^2 + (430-720\eta-45e^2)\theta^4 \right] \tag{86a}$$

For $\dot{\Omega}$

$$\tilde{n}\theta \; \frac{3}{32} \frac{J_2^2 a_e^4}{p^4} \left[ 4-24\eta-9e^2 + (-40+72\eta+5e^2)\theta^2 \right] \tag{87a}$$

Suppose now that we wish to use the smoothed elements $\bar{n}$, $a'$, $e'$, $I'$, $\omega'$ and $\Omega'$ as defined in Equation (55). If we enter the $J_2$ terms for $\dot{\omega}$ and $\dot{\Omega}$ in Table VII with

$$p' = a' (1-e'^2)$$

$$\eta' = (1-e'^2)^{1/2}$$

$$\theta' = \cos I'$$

then the $J_2^2$ terms in Equation (86) and (87) will be changed by

$$\delta\dot{\omega} = \bar{n} \left(-\frac{3}{4}\frac{J_2 a_e^2}{p^2}(1-5\theta^2)\right)\left[-2\frac{\delta a}{a} + \frac{4e\ \delta e}{\eta^2} + \frac{10 \sin I\theta\ \delta I}{(1-5\theta^2)}\right] \tag{92}$$

$$\delta\dot{\Omega} = \bar{n}\ \theta \left(-\frac{3}{2}\frac{J_2 a_e^2}{p^2}\right)\left[-2\frac{\delta a}{a} + \frac{4e\ \delta e}{1-e^2} - \frac{\sin I\ \delta I}{\theta}\right] \tag{93}$$

where the $\delta\epsilon_i$ are related to the $\Delta\epsilon_i$ of Equation (55) by

$$\delta\epsilon_i = \epsilon_i - \epsilon_i' = -\Delta\epsilon_i$$

The results are

<u>For $\dot{\omega}$</u>

$$\bar{n}\frac{3}{128}\frac{J_2^2 a_e^4}{p^4}\left[-42-57e^2 + (-20+382e^2)\theta^2 + (190-525e^2)\theta^4\right] \tag{86b}$$

For $\dot{\Omega}$

$$\bar{n}\theta \frac{3}{32} \frac{J_2^2 a_e^4}{p^4} \left[ -25e^2 + (-4+53e^2)\theta^2 \right] \tag{87b}$$

These terms differ from those given by Merson[7] because he employs $u$ as the argument of the secular terms rather than $M$ (in the form $\bar{n}t$ ), i.e., his rates apply for a nodal rather than an anomalistic frequency and may be obtained by subtracting $\frac{\dot{\omega}^2}{\bar{n}}$ and $\frac{\dot{\omega}\dot{\Omega}}{\bar{n}}$ from Equations (86b) and (87b).

Now since from Equation (55),

$$\omega' - \omega = -\frac{3}{4} \frac{J_2 a_e^2}{p^2} (1-5\theta^2)\ (v-M) \approx \frac{\dot{\omega}}{n}\ (v-M)$$

$$\Omega' - \Omega = -\frac{3}{2} \frac{J_2 a_e^2}{p^2} \theta\ (v-M) \approx \frac{\dot{\Omega}}{n}\ (v-M)$$

if we define the increase in mean anomaly to be

$$\Delta M = \bar{n}\ \Delta t \tag{94}$$

and we define $\Delta v$ to be the related quantity

$$\Delta v = \Delta M + (v-M) \tag{95}$$

(note that $\Delta v$ is not zero at the epoch, unless epoch is defined to be at perigee or apogee ) we have

$$\omega' = \omega_o + \dot{\omega}\,\Delta t + \frac{\dot{\omega}}{\bar{n}}\,(v-M) \tag{96}$$

$$= \omega_o' + \frac{\dot{\omega}}{\bar{n}}\,\Delta v$$

and similarly

$$\Omega' = \Omega_o' + \frac{\dot{\Omega}}{\bar{n}}\,\Delta v \tag{97}$$

where $\frac{\dot{\omega}}{\bar{n}}$ and $\frac{\dot{\Omega}}{\bar{n}}$ are obtained from Table VII as modified by Equations (86b) and (87b). It remains to determine $a'$ (and, thereby $p'$). Since

$$a' = a\left(1 + \frac{1}{2}\,\frac{J_2 a_e^2}{p^2}\,\eta\,(1-3\theta^2)\right)$$

Equation (88) becomes

$$a' = \mu^{1/3}\,\bar{n}^{-2/3}\left\{1 + \frac{1}{64}\,\frac{J_2^2 a_e^4}{p^4}\,\eta\left[10-4\eta-25e^2 + (-60+24\eta+90e^2)\theta^2 + (130-36\eta-25e^2)\theta^4\right] - \frac{15}{64}\,\frac{J_4 a_e^4}{p^4}\,\eta\,e^2\left[3-30\theta^2 + 35\theta^4\right]\right\}$$

$$= \mu^{1/3}\,\bar{n}^{-2/3}\left\{1 + S\eta\left(\frac{4j}{3}\,\frac{1-e^2}{e^2} - 2\right) + \left[J_2^2 \text{ term above}\right]\right\} \tag{88b}$$

$$(n > 2)$$

Thus, to the first order, a' may be computed from $\overline{n}$ using the Keplerian relationship and no special provisions are required for computing p'.

The non-conservative perturbations are sometimes represented by polynomials in time. If, for example

$$M = M + n\,\Delta t + \frac{\dot{n}}{2}\,\Delta t^2 + \frac{\ddot{n}}{6}\,\Delta t^3 \tag{98}$$

$$\overline{n} = n + \dot{n}\,\Delta t + \frac{\ddot{n}}{2}\,\Delta t^2 \tag{99}$$

then the coefficients for a'

$$a' = a + \dot{a}\,\Delta t + \frac{\ddot{a}}{2}\,\Delta t^2 \tag{100}$$

are easily obtained from Equation (88b) as

$$a = \mu^{1/3}\,n^{-2/3} \tag{101}$$

$$\dot{a} = -\frac{2}{3}\,\frac{\dot{n}}{n}\,a \tag{102}$$

$$\frac{\ddot{a}}{2} = \left(\frac{5}{9}\left(\frac{\dot{n}}{n}\right)^2 - \frac{1}{3}\,\frac{\ddot{n}}{n}\right) a \tag{103}$$

A similar polynomial can be employed for e'

$$e' = e + \dot{e}\,\Delta t + \frac{\ddot{e}}{2}\,\Delta t^2 \tag{104}$$

The time dependent terms may be obtained empirically, or from the approximation that perigee height remains constant

$$\frac{d}{dt}\, a'\,(1-e') = 0$$

which yields

$$\dot{e} = \frac{1-e}{a}\,\dot{a} = -\,\frac{2}{3}\,\frac{1-e}{n}\,\dot{n} \tag{105}$$

$$\frac{\ddot{e}}{2} = \frac{1-e}{a}\left(\frac{\ddot{a}}{2} - \frac{\dot{a}^2}{a}\right) = \frac{1-e}{n}\left(\frac{1}{9}\,\frac{\dot{n}^2}{n} - \frac{1}{3}\,\ddot{n}\right) \tag{106}$$

The inclination is usually assumed constant, although it may be represented by a similar series. Where these secular perturbations on $\bar{n}$, $a'$, and $e'$ exist, they must be reflected in the motion of node and perigee. The variations in $\bar{n}$ are already incorporated in Equations (96) and (97). It is usually adequate to carry the additional variations due to $a'$, $e'$, and $I'$ to the first order in $J_2$; the necessary derivatives of $\frac{\dot{\omega}}{\bar{n}}$ and $\frac{\dot{\Omega}}{\bar{n}}$ may be obtained from Equations (92) and (93). . The resulting equations are

$$\begin{pmatrix} \omega' \\ \Omega' \end{pmatrix} = \begin{pmatrix} \omega_o \\ \Omega_o \end{pmatrix} + \begin{pmatrix} \frac{\dot{\omega}}{\bar{n}} \\ \frac{\dot{\Omega}}{\bar{n}} \end{pmatrix} \Delta v \left( 1 - 2\,\frac{\delta a}{a} + \frac{4e}{1-e^2}\,\delta e + \begin{pmatrix} \frac{10 \sin I\theta}{1-5\theta^2} \\ -\,\frac{\sin I}{\theta} \end{pmatrix} \delta I \right) \tag{107}$$

where the $\delta$'s represent the change from epoch, or

$$\begin{pmatrix} \omega' \\ \\ \Omega' \end{pmatrix} = \begin{pmatrix} \omega_o \\ \\ \Omega_o \end{pmatrix} + \begin{pmatrix} \frac{\dot{\omega}}{n} \\ \\ \frac{\dot{\Omega}}{n} \end{pmatrix} \Delta v \; (1+\alpha) \tag{108}$$

where (using Equations (102) and (103) )

$$\alpha = \left( \frac{4}{3} \frac{\dot{n}}{n} + \frac{4e}{1-e^2} \dot{e} + \begin{pmatrix} \frac{10 \sin I\theta}{1-5\theta^2} \\ \\ - \frac{\sin I}{\theta} \end{pmatrix} \dot{I} \right) \Delta t$$

$$+ \left( - \frac{10}{9} \left(\frac{\dot{n}}{n}\right)^2 + \frac{2}{3} \frac{\ddot{n}}{n} + \frac{2e}{1-e^2} \ddot{e} + \begin{pmatrix} \frac{10 \sin I\theta}{1-5\theta^2} \\ \\ - \frac{\sin I}{\theta} \end{pmatrix} \frac{\ddot{I}}{2} \right) \Delta t^2 \tag{109}$$

or, for constant perigee height and inclination (see Equations (105) and (106) ),

$$\alpha = \frac{4(1-e)}{3(1+e)} \frac{\dot{n}}{n} \Delta t + \left( \frac{-10-6e}{9(1+e)} \left(\frac{\dot{n}}{n}\right)^2 + \frac{2(1-e)}{3(1+e)} \frac{\ddot{n}}{n} \right) \Delta t^2 \tag{109a}$$

In some cases it is desirable to express the secular terms as functions of time rather than as functions of $\Delta v$. It is possible to expand Equations (108), (109) and (109a) using Equation (95), and by separating like powers of $\Delta t$ to obtain

$$\begin{pmatrix} \omega' \\ \\ \Omega' \end{pmatrix} = \begin{pmatrix} \omega_o \\ \\ \Omega_o \end{pmatrix} + \begin{pmatrix} \dot{\omega} \\ \\ \dot{\Omega} \end{pmatrix} \left( \Delta t + \frac{v-M}{n} \right) + \begin{pmatrix} \frac{\ddot{\omega}}{2} \\ \\ \frac{\ddot{\Omega}}{2} \end{pmatrix} \Delta t^2 + \begin{pmatrix} \frac{\dddot{\omega}}{6} \\ \\ \frac{\dddot{\Omega}}{6} \end{pmatrix} \Delta t^3 + \ldots \tag{110}$$

where

$$\begin{pmatrix} \dot{\omega} \\ \\ \dot{\Omega} \end{pmatrix} = \begin{pmatrix} \frac{\dot{\omega}}{\bar{n}} \\ \\ \frac{\dot{\Omega}}{\bar{n}} \end{pmatrix} \bar{n} \tag{111}$$

$$\begin{pmatrix} \frac{\ddot{\omega}}{2} \\ \\ \frac{\ddot{\Omega}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\dot{\omega}}{\bar{n}} \\ \\ \frac{\dot{\Omega}}{\bar{n}} \end{pmatrix} \left[ \frac{11}{6}\,\dot{n} + \frac{4e}{1-e^2}\, n\dot{e} + \begin{pmatrix} \frac{10 \sin I\theta}{1-5\theta^2} \\ \\ -\frac{\sin I}{\theta} \end{pmatrix} n\dot{I} \right] \tag{112}$$

$$= \begin{pmatrix} \frac{\dot{\omega}}{\bar{n}} \\ \\ \frac{\dot{\Omega}}{\bar{n}} \end{pmatrix} \left[ \frac{11 - 5e}{6(1+e)} \quad \dot{n} \right] \tag{112a}$$

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13. ABSTRACT

The periodic position and velocity perturbations of an artificial earth satellite are developed to the first order for all $J_n$, based on the theory by Brouwer as extended by Giacagliz. An explicit formulation is also provided for the subset $J_2$, $J_3$, $J_4$. The use of a position and velocity formulation circumvents the equatorial and circular orbit singularities found in conventional developments. The definition of the mean elements of the theory is modified to reduce the complexity of the position perturbations, as suggested by Merson's Theory, and the resulting changes to the secular terms are developed. In order to facilitate an empirical correction for drag, the observed mean motion is introduced as a mean element in place of the semi-major axis.

| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| SATELLITE ORBITAL MOTION | | | | | | |
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